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An inequality for degree sequences

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Abstract

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Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a simple graph and suppose p is a positive integer. We show that $(\sum_{i=1}^n d_i^{1/p})^p \geq \sum_{i=1}^n d_i^p$. Related 'real' inequalities, i.e., not graph-dependent, are analyzed.

1. Introduction

The main objective of the present paper is to prove the following theorem.

Theorem 1. *If $p \geq 1$ is an integer and $d_1, d_2, \ldots, d_n$ is the degree sequence of a simple graph, then*

$$\left(\sum_{i=1}^n d_i^{1/p}\right)^p \geq c \sum_{i=1}^n d_i^p, \tag{1}$$

for $c = 1$ while (1) *is not always true for any constant $c > 1$. Equality holds in* (1) *iff $p = 1$ or the graph is empty.*

In addition, we analyze related 'real', i.e., not graph-dependent, inequalities.

Some of the lemmas are interesting for their own sake. Lemma 3 generalizes Chebyshev's inequality; Lemma 4, in some cases, sharpens the inequality of arithmetic and harmonic means.

For comparison we investigate what $c = c_{n,p}$ makes (1) true if we drop the requirement of having a degree sequence. An integral inequality analogous to (1)

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is derived as well. The $p=2$ case of the theorem arose from probabilistic analysis of some graph bisection problems.

2. Some lemmas

Lemma 1. *Suppose $p \geqslant 1$ is an integer. If the sequences $d_i \geqslant 0$ and $D_i \geqslant 0$ satisfy* (1) *then all their convex combinations satisfy* (1).

Proof. Take a convex combination $\lambda d_i + (1-\lambda)D_i$ for $0 \leqslant \lambda \leqslant 1$. Then

$$\begin{aligned}
&\left\{\sum_{i=1}^{n} (\lambda d_i + (1-\lambda)D_i)^{1/p}\right\}^p \\
&\quad \geqslant \left\{\sum_{i=1}^{n} \lambda d_i^{1/p} + (1-\lambda)D_i^{1/p}\right\}^p \\
&\quad = \sum_{k=0}^{p} \binom{p}{k} \lambda^k (1-\lambda)^{p-k} \left(\sum_{i=1}^{n} d_i^{1/p}\right)^k \left(\sum_{i=1}^{n} D_i^{1/p}\right)^{p-k} \\
&\quad \geqslant c \sum_{k=0}^{p} \binom{p}{k} \lambda^k (1-\lambda)^{p-k} \left(\sum_{i=1}^{n} d_i^p\right)^{k/p} \left(\sum_{i=1}^{n} D_i^p\right)^{(p-k)/p} \\
&\quad = c \sum_{k=1}^{p-1} \binom{p}{k} \lambda^k (1-\lambda)^{p-k} \left(\sum_{i=1}^{n} (d_i^k)^{p/k}\right)^{k/p} \left(\sum_{i=1}^{n} (D_i^{p-k})^{p/(p-k)}\right)^{(p-k)/p} \\
&\qquad + c\lambda^p \left(\sum_{i=1}^{h} d_i^p\right) + c(1-\lambda)^p \left(\sum_{i=1}^{n} D_i^p\right) \\
&\quad \geqslant c \sum_{k=0}^{p} \binom{p}{k} \lambda^k (1-\lambda)^{p-k} \sum_{i=1}^{n} d_i^k D_i^{p-k} = c \sum_{i=1}^{n} (\lambda d_i + (1-\lambda)D_i)^p.
\end{aligned}$$

The first inequality is Jensen's for the convex function $-x^{1/p}$ (Theorem 86 in [1]); the second inequality follows from the hypothesis on d_i and D_i; and the third inequality is Hölder's [1, (2.8.3)]. $\square$

Lemma 2 (Koren [2]). *To each degree sequence $d_1, d_2, \ldots, d_n$ of a simple graph on n vertices, assign the point $(d_1, d_2, \ldots, d_n) \in \mathbb{R}^n$. Let E_n denote the set of extremal points of the convex hull of the set of points in $\mathbb{R}^n$ assigned to degree sequences. Let F_n denote the following recursively defined sets of sequences: $F_1 = \{\alpha\}$, where α is the one term sequence of* 0, *$\beta = (b_1, b_2, \ldots, b_{n+1}) \in F_{n+1}$ iff either $b_1 = n$ and $(b_2 - 1, \ldots, b_{n+1} - 1) \in F_n$ or $b_{n+1} = 0$ and $(b_1, b_2, \ldots, b_n) \in F_n$. Obviously, F_n contains 2^{n-1} sequences. Then the elements of E_n are given by all permutations of F_n.* $\square$

Definition. We say the sequences a_i and b_i are *similar* if $a_i > a_j$ implies $b_i > b_j$.

Lemma 3. *Suppose $a_i \geqslant 0$ and $b_i \geqslant 0$ $(i = 1, \ldots, n)$ are similar and $f(x, y) \in C^2([0, \infty) \times [0, \infty))$ with $(\partial^2/\partial x^2)f \geqslant 0$ and $(\partial^2/\partial x\, \partial y)f \geqslant 0$ in that domain. Then*

$$\sum_{i=1}^{n} f\left(\frac{1}{n}\sum_{j=1}^{n} a_j, b_i\right) \leqslant \sum_{i=1}^{n} f(a_i, b_i).$$

If $(0, 0)$ is not allowed to be an (a_i, b_i), then the lemma holds even if $(0, 0)$ is not in the domain of $f(x, y)$.

Proof. Suppose $0 \leqslant a < A$ and $0 \leqslant b < B$. We claim

$$D(a, b, A, B) = f(a, b) + f(A, B) - f(a, B) - f(A, b) \geqslant 0. \tag{2}$$

It follows from $D(a, b, A, B) = \int_a^A \int_b^B (\partial^2/\partial x\, \partial y)f \, dx \, dy$.

Notice that (2) holds if $a = A$ or $b = B$, since in this case $D(a, b, A, B) = 0$. To finish the proof of the lemma, use (2) and $D(a, b, A, B) = D(A, B, a, b)$ to see that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} D(a_i, b_i, a_j, b_j) \geqslant 0 \tag{3}$$

since a_i and b_i are similar. By the definition of D we have

$$2n \sum_{i=1}^{n} f(a_i, b_i) \geqslant 2 \sum_{i=1}^{n} \sum_{j=1}^{n} f(a_i, b_j) \geqslant 2n \sum_{j=1}^{n} f\left(\frac{1}{n}\sum_{i=1}^{n} a_i, b_j\right).$$

The second inequality is Jensen's for $f(x, b_j)$, which is a convex function of x for all b_j, since $(\partial^2/\partial x^2)f \geqslant 0$. The case of exceptional $(0, 0)$ is left to the reader. □

Remark. Lemma 3 generalizes Chebyshev's inequality, Theorem 43 in [1]. It follows from setting $f(x, y) = xy$.

Lemma 4. *Suppose $f > 0$. Under the conditions of Lemma 3 together with $f > 0$ and $(\partial^2/\partial y^2)f \geqslant 0$, we have*

$$\frac{n}{\sum_{i=1}^{n} f(a_i, b_i)} \leqslant \frac{1}{f\left(\frac{1}{n}\sum_{i=1}^{n} a_i, \frac{1}{n}\sum_{i=1}^{n} b_i\right)}.$$

Proof. Since $(\partial^2/\partial y^2) \geqslant 0$, $f(c, y)$ is a convex function of y for all c. Let $c = \sum a_j/n$ and apply Jensen's inequality:

$$\frac{1}{f\left(\frac{1}{n}\sum_{i=1}^{n} a_i, \frac{1}{n}\sum_{i=1}^{n} b_i\right)} \geqslant \frac{n}{\sum_{i=1}^{n} f(c, b_i)} \geqslant \frac{n}{\sum_{i=1}^{n} f(a_i, b_i)},$$

where the latter inequality holds by Lemma 4. □

Remark. Lemma 4 sharpens the inequality of harmonic and arithmetic means for the numbers $1/f(a_i, b_i)$, whenever

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{f(a_i, b_i)} \geqslant \frac{1}{f\left(\frac{1}{n}\sum_{i=1}^{n} a_i, \frac{1}{n}\sum_{i=1}^{n} b_i\right)}.$$

This is the case in the proof of Theorem 1 for $p>2$. For $p=2$, it turns into the inequality of harmonic and arithmetic means.

Lemma 5. *Suppose $m \geqslant 1$ is an integer and $g(x, y)=\sum_{t=0}^{m} x^i y^{m-i}$. For $f(x, y)=1/g(x, y)$, we have*

$$f(x, y)>0, \qquad \frac{\partial^2}{\partial x\, \partial y} f(x, y) \geqslant 0, \qquad \frac{\partial^2}{\partial x^2} f(x, y) \geqslant 0, \qquad \frac{\partial^2}{\partial y^2} f(x, y) \geqslant 0$$

in $[0, \infty) \times [0, \infty)-\{(0,0)\}$. Consequently, all the conditions of Lemma 4 hold.

Proof. First we prove $(\partial^2/\partial x\, \partial y) f(x, y) \geqslant 0$. Since

$$\frac{\partial^2}{\partial x\, \partial y} f=\frac{2 \frac{\partial g}{\partial x} \frac{\partial g}{\partial y}-g \frac{\partial^2 g}{\partial x\, \partial y}}{g^3},$$

it is sufficient to prove that $Q=2(\partial g/\partial x)(\partial g/\partial y)-g(\partial^2/\partial x\, \partial y) \geqslant 0$. We are going to show that all the coefficients of Q are nonnegative. We use the notation $[x^i y^j] G$ for the coefficient of $x^i y^j$ in G. It is easy to see that in this case

$$\begin{aligned}[x^i y^{2m-2-i}] Q= & \sum_{t=\max(0, i-m+1)}^{\min(i, m-1)}(t+1)(m-i+t) \\ & -\sum_{t=\max(0, i-m+2)}^{\min(i, m)}(i-t+1)(m-i+t-1).\end{aligned}$$

Q is symmetric in x and y, therefore, it is sufficient to show that

$$[x^i y^{2m-2-i}] Q \geqslant 0,$$

for $0 \leqslant i \leqslant m-1$.

This observation simplifies the evaluation of our sums with complicated bounds. We get after some algebra

$$\begin{aligned}[x^i y^{2m-2-i}] Q & =2 \sum_{t=0}^{i}(t+1)(m-i+t)-\sum_{t=0}^{i}(i-t+1)(m-i+t-1) \\ & =(i+1)[i/2+m i/2+m+1]>0.\end{aligned}$$

The bounds for summation are correct for $i=m-1$ as well, since the added term is zero.

By the symmetry of f in x and y, $(\partial^2/\partial x^2) f \geqslant 0$ implies $(\partial^2/\partial y^2) f \geqslant 0$. We prove the first inequality. Since

$$\frac{\partial^2 f}{\partial x^2}=\frac{2(\partial g/\partial x)^2-g(\partial^2/\partial x^2)g}{g^3},$$

it is sufficient to prove $R=2(\partial g/\partial x)^2-g(\partial^2/\partial x^2)g\geqslant 0$. We remark that R is no longer symmetric in x and y. It is easy to see that

$$[x^iy^{2m-2-i}]R=\sum_{t=\max(0,i+1-m)}^{\min(i,m-1)}2(t+1)(i-t+1)-\sum_{t=\max(0,i+2-m)}^{\min(i,m)}(i-t+1)(i-t+2).$$

For the evaluation of the summations on the right-hand side, we have three different cases: $0\leqslant i\leqslant m-2$, $i=m-1$ and $m\leqslant i\leqslant 2m-2$. In the first case, $[x^iy^{2m-2-i}]R=0$, the second and third cases result in the same summation, similar to the case of the mixed second derivative:

$$\sum_{t=i+1-m}^{m-1}2(t+1)(i-t+1)-\sum_{t=i+2-m}^{m}(i-t+1)(i-t+2)$$
$$=(m+1)[-i^2+i3(m-1)-2m^2+5m-2].$$

This is a quadratic expression in i with a negative leading coefficient, therefore, it is non-negative in the interval $[m-1,2(m-1)]$ since it is nonnegative in $m-1$ and $2(m-1)$. □

3. Proof of Theorem 1

We prove the theorem by induction on p.

For $p=1$, (1) becomes an identity. We show the inductive step by induction on n where the base case of $n=1$ is an identity for any p.

By Lemma 1, it suffices to prove (1) for the degree sequences of E_{n+1}; the extremal points of the convex hull of degree sequences in $\mathbb{R}^{n+1}$. Lemma 2 describes, up to permutation, the extremal degree sequences:

$$(d_1,d_2,\ldots,d_n,0)\quad\text{where }(d_1,d_2,\ldots,d_n)\in E_n,$$

or

$$(d_1+1,d_2+1,\ldots,d_n+1,n)\quad\text{where }(d_1,d_2,\ldots,d_n)\in E_n.$$

If (1) holds for $(d_1,d_2,\ldots,d_n)$, then it automatically holds for $(d_1,d_2,\ldots,d_n,0)$ since both sides of (1) remain unchanged. Suppose (1) holds for $(d_1,d_2,\ldots,d_n)$ and set

$$A=n^p+\sum_{i=1}^{n}(d_i+1)^p-\sum_{i=1}^{n}d_i^p=n^p+n+\sum_{k=1}^{p-1}\binom{p}{k}\sum_{i=1}^{n}d_i^k,$$

and

$$B=\left(n^{1/p}+\sum_{i=1}^{n}(d_i+1)^{1/p}\right)^p-\left(\sum_{i=1}^{n}d_i^{1/p}\right)^p.$$

Since summing, $B \geqslant A$ and (1) give the desired inequality for $(d_1+1, d_2+1, \ldots, d_n+1, n)$, it suffices to prove $B \geqslant A$.

With $g(x,y)=\sum_{i=0}^{p-1} x^i y^{p-1-i}$ observe that

$$B=\left[\left(n^{1/p}+\sum_{i=1}^{n}(d_i+1)^{1/p}\right)-\left(\sum_{i=1}^{n} d_i^{1/p}\right)\right]\cdot g\left(n^{1/p}+\sum_{i=1}^{n}(d_i+1)^{1/p}, \sum_{i=1}^{n} d_i^{1/p}\right)$$

and similarly

$$1=[(d_i+1)^{1/p}]^p-[d_i^{1/p}]^p=[(d_i+1)^{1/p}-d_i^{1/p}]\cdot g((d_i+1)^{1/p}, d_i^{1/p})$$

so that

$$B=\left(n^{1/p}+\sum_{i=1}^{n}\frac{1}{g((d_i+1)^{1/p}, d_i^{1/p})}\right)\cdot g\left(n^{1/p}+\sum_{i=1}^{n}(d_i+1)^{1/p}, \sum_{i=1}^{n} d_i^{1/p}\right)$$

while the last factor above is equal to

$$\sum_{t=0}^{p-1}\sum_{l=0}^{t} n^{l/p}\binom{t}{l}\left(\sum_{i=1}^{n}(d_i+1)^{1/p}\right)^{t-l}\left(\sum_{i=1}^{n} d_i^{1/p}\right)^{p-1-t}$$
$$=\sum_{l=0}^{p-1} n^{l/p}\sum_{t=l}^{p-1}\binom{t}{l}\left(\sum_{i=1}^{n}(d_i+1)^{1/p}\right)^{t-l}\left(\sum_{i=1}^{n} d_i^{1/p}\right)^{p-1-t}. \tag{4}$$

We give now the terms of B which majorize the terms of A. Lemmas 4 and 5 applied to the similar sequences $n(d_i+1)^{1/p}$ and $nd_i^{1/p}$ give

$$n \leqslant \frac{1}{n^{p-1}}\cdot\sum_{i=1}^{n}\frac{1}{g((d_i+1)^{1/p}, d_i^{1/p})}\cdot g\left(\sum_{i=1}^{n}(d_i+1)^{1/p}, \sum_{i=1}^{n} d_i^{1/p}\right)$$

so that

$$n^p \leqslant \sum_{i=1}^{n}\frac{1}{g((d_i+1)^{1/p}, d_i^{1/p})}\cdot\sum_{t=0}^{p-1}\left(\sum_{i=1}^{n}(d_i+1)^{1/p}\right)^{t}\left(\sum_{i=1}^{n} d_i^{1/p}\right)^{p-1-t}$$

where the first factor is part of the first factor of B and the last factor is the $l=0$ summand of (4). Also

$$n \leqslant n^{1/p}\cdot n^{p-1/p}$$

where the last factor is the $l=p-1$ summand of (4). Finally, for fixed $1 \leqslant k \leqslant p-1$, we have

$$d_i^{1/p-k} \leqslant n^{k/p(p-k)} d_i^{1/p}, \quad \text{since } d_i < n$$

and

$$\binom{p}{k}=\sum_{t=k-1}^{p-1}\binom{t}{k-1}$$

so that, by our inductive hypothesis applied to $p-k$,

$$\begin{aligned}\binom{p}{p-k}\sum_{i=1}^{n} d_i^{p-k} &\leqslant \binom{p}{k}\Big(\sum_{i=1}^{n} d_i^{1/(p-k)}\Big)^{p-k} \quad \text{(by induction)}\\ &\leqslant n^{k/p}\cdot\sum_{t=k-1}^{p-1}\binom{t}{k-1}\Big(\sum_{i=1}^{n} d_i^{1/p}\Big)^{p-k}\\ &= n^{1/p}\cdot n^{k-1/p}\sum_{t=k-1}^{p-1}\binom{t}{k-1}\Big(\sum_{i=1}^{n} d_i^{1/p}\Big)^{(p-1)-(k-1)},\end{aligned}$$

where the first factor is the other part of the first factor of B (used for $k=1$) and the product of the last three factors is the $l=k-1$ summand of (4).

We note that, for fixed $p \geqslant 1$ the sharpness of Theorem 1 is given by the graphs formed from a complete k-graph together with $n-k$ isolated vertices where $k, n \to \infty$.

We leave the case of equality to the reader. □

4. Related real inequalities

Theorem 2. *Any sequence d_i ($d_i \in [0, n-1]$) satisfies inequality* (1) *with $c = (n-1)^{1-p}$. This is the best possible constant in the inequality for integer p.*

Proof. By Lemma 1, for integer $p \geqslant 1$, the best constant is determined by the minimum ratio of $(\sum_{i=1}^{n} d_i^{1/p})^p / \sum_{i=1}^{n} d_i^p$, where the sequence d_i belongs to the vertices of the cube $[0, n-1]^n$ with the origin deleted. □

Theorem 3. *If integer $p \geqslant 1$, $n_i > 0$, $(i = 1, 2, \ldots, m)$, $n = \sum_{i=1}^{m} n_i$ and $l = \lfloor (m+1)/2 \rfloor$, $k = \lfloor (m-2)/2 \rfloor$, then*

$$\Bigg[\sum_{j=1}^{l} n_j\Big(n-1-\sum_{t=m-j+2}^{m} n_t\Big)^{1/p} + \sum_{j=m-k}^{m} n_j\Big(\sum_{t=1}^{m-j+1} n_t\Big)^{1/p}\Bigg]^p$$
$$\geqslant \sum_{j=1}^{l} n_j\Big(n-1-\sum_{t=m-j+2}^{m} n_t\Big)^p + \sum_{j=m-k}^{m} n_j\Big(\sum_{t=1}^{m-j+1} n_t\Big)^p.$$

Proof. An explicit description of the extreme degree sequences of F_n was given in Theorem 2 of [2]. The vertices of F_n are the sequences

$$(\overbrace{b_1, \ldots, b_1}^{n_1}, \ldots, \overbrace{b_m, \ldots, b_m}^{n_m}),$$

where $b_1 > b_2 > \cdots > b_m$ and

$$\begin{aligned} b_1 &= n-1, & b_m &= n_1,\\ b_2 &= n-1-n_m, & b_{m-1} &= n_1+n_2,\\ &\vdots & &\vdots\\ b_l &= n-1-n_m-\cdots-n_{m-l+2}, & b_{m-k} &= n_1+n_2+\cdots+n_{k+1}.\end{aligned}$$

Theorem 1 holds for the extreme degree sequences as well, as applying (1) to the sequence b_i shows. □

Theorem 4. *If $p \geqslant 1$ and integer, $f \geqslant 0$, $f \in L^1[0, 1]$ with $\int_0^1 f(x)\,dx = 1$, then*

$$\left(\int_0^1 f(x)\left\{\int_0^{1-x} f(t)\,\mathrm{d}t\right\}^{1/p} \mathrm{d}x\right)^p \geqslant \int_0^1 f(x)\left\{\int_0^{1-x} f(t)\,\mathrm{d}t\right\}^p \mathrm{d}x. \tag{5}$$

Proof. We note that if (6) holds for f_n and $\|f_n - f\|_{L^1} \to 0$, then (5) holds for f. Since $C[0, 1]$ is dense in $L^1[0, 1]$, it suffices to prove (6) for $f \in C[0, 1]$. Define a sequence of step functions by $F_n(x) = (m_n/n)n_i$ for $x \in [(i-1)/m_n, i/m_n)$, where $i = 1, \ldots, m_n$, such that:

- $n_i > 0$ integer, $\sum_{i=1}^{m_n} n_i = n$;
- $F_n(x) \to f(x)$ uniformly, when $n \to \infty$.

After applying Theorem 3 to n_i, multiplying both sides of the inequality by n^{-p-1} and taking limits we obtain

$$\left[\int_0^{\frac{1}{2}} f(x)\left\{1 - \int_{1-x}^1 f(t)\,\mathrm{d}t\right\}^{1/p} \mathrm{d}x + \int_{\frac{1}{2}}^1 f(x)\left\{\int_0^{1-x} f(t)\,\mathrm{d}t\right\}^{1/p} \mathrm{d}x\right]^p$$
$$\geqslant \int_0^{\frac{1}{2}} f(x)\left\{1 - \int_{1-x}^1 f(t)\,\mathrm{d}t\right\}^p \mathrm{d}x + \int_{\frac{1}{2}}^1 f(x)\left\{\int_0^{1-x} f(t)\,\mathrm{d}t\right\}^p \mathrm{d}x$$

which is equivalent to (5). □

We conjecture that Theorems 1, 3 and 4 hold for all real $p \geqslant 1$. Actually, we have an alternative proof of Lemma 1 which works for real $p \geqslant 1$, and likely Lemma 5 is doable for $g(x, y) = (x^p - y^p)/(x - y)$. We do not see how to drop the induction by p in the proof of Theorem 1.

References

[1] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities (Univ. Press, Cambridge, 2nd edition, 1952).

[2] M. Koren, Extreme degree sequences of simple graphs, J. Combin. Theory Ser. B 15 (1973) 213–224.